STRICTLY N-FINITE VARIETIES OF HEYTING ALGEBRAS

T. HYTTINEN, M. MARTINS, T. MORASCHINI, AND D. E. QUADRELLARO

Abstract. For any $n < \omega$ we construct an infinite $(n+1)$-generated Heyting algebra whose n-generated subalgebras are of cardinality $\leq m_n$ for some positive integer m_n. From this we conclude that for every $n < \omega$ there exists a variety of Heyting algebras which contains an infinite $(n+1)$-generated algebra, but which contains only finite n-generated algebras. For the case $n = 2$ this provides a negative answer to a question posed by G. Bezhanishvili and R. Grigolia in [**bezhanishvili2005locally**].

1. Introduction

A *Heyting algebra* $(H, \wedge, \vee, \to, 0, 1)$ is a bounded distributive lattice with a binary operation $\to$ such that

$$a \wedge b \leq c \Longleftrightarrow a \leq b \to c,$$

for every $a, b, c \in H$ [**BaDw74**, **ChZa97**, **esakia**, **RaSi70**]. Heyting algebras appear naturally in many areas of mathematics. For instance, the lattice of open sets of a topological space forms a Heyting algebra. The subobject classifier of a topos can also be endowed with the structure of a Heyting algebra. Lastly, every distributive algebraic lattice is a Heyting algebra.

In this paper, we will focus on finitely generated Heyting algebras. We recall that an algebra A is said to be *n-generated* when there is a subset $X \subseteq A$ of size $\leq n$ such that the least subalgebra of A containing X is A itself. Accordingly, we say that A is *finitely generated* when it is n-generated for some $n < \omega$. A class of similar algebras that can be axiomatised by (universally quantified) equations is called a *variety*. Examples of varieties include the class of all Heyting algebras, as well as that of all Boolean algebras. A variety is said to be *n-finite* when its n-generated members are finite, and *locally finite* when it is n-finite for every $n < \omega$. We call a variety *strictly n-finite* if it is n-finite, but not $(n+1)$-finite.

Dual characterisations of finitely generated Heyting algebras were obtained in [**esakia1977criterion**] (see also [**bezhanishvili2006lattices**]), while locally finite varieties of Heyting algebras were studied by G. Bezhanishvili and R. Grigolia in [**bezhanishvili2005locally**]. In the same paper, they raise the following question [**bezhanishvili2005locally**]: is it true that a variety of Heyting algebras is locally finite iff it is 2-finite? While this holds in the restrictive context of varieties of Heyting algebras of width two [**benjamins2020locally**], in this paper we establish that for any $n < \omega$ there exists an infinite $(n+1)$-generated Heyting algebra H_n whose n-generated subalgebras are of size $\leq m_n$ for some $m_n < \omega$ (Theorem 21). It follows that the variety generated by H_n fails to be locally finite, although it is

Date: February 21, 2024.

1991 *Mathematics Subject Classification.* 03C05, 06D20, 08B99.

The second author was supported by the research grant PREDOCS-UB 2021 funded by the University of Barcelona. The third author was supported by the *Beatriz Galindo* grant BEAGAL-18/00040 funded by the Ministry of Science and Innovation of Spain. The fourth author was supported by grant 336283 of the Academy of Finland and Research Funds of the University of Helsinki.

n-finite. For $n = 2$ this provides a negative answer to Bezhanishvili and Grigolia's question.

This result was first established in 2020, although it never appeared in print [**martins2023locally**]. Independently, the first and fourth authors discovered an alternative simpler proof in 2023 [**hyttinen2023varieties**]. To make the result available, we decided to publish the latter together.

2. Esakia Duality

In this section, we will review the Esakia duality [**Es74**, **esakia**] between Heyting algebras and Esakia spaces. We start by fixing some notation: whenever $(X, \leq)$ is a poset and $Y \subseteq X$ we let

$$Y^{\uparrow} = \{x \in X \mid \exists y \in Y \text{ and } y \leq x\} \quad \text{and} \quad Y^{\downarrow} = \{x \in X \mid \exists y \in Y \text{ and } y \geq x\}.$$

For $x \in X$, we write $x^{\uparrow}$ and $x^{\downarrow}$ for the sets $\{x\}^{\uparrow}$ and $\{x\}^{\downarrow}$ respectively. A subset $Y \subseteq X$ is said to be an *upset* when $U = U^{\uparrow}$. We write $\mathsf{Up}(X)$ for the set of upsets of a poset X. Also, given any subset $Y \subseteq X$ we write Y^c for its complement $X \smallsetminus Y$.

We recall that an *Esakia space* is a triple $X = (X, \tau, \leq)$, where (X, τ) is a compact topological space and $(X, \leq)$ a poset satisfying the following conditions:

(i) *Priestley separation axiom*: For all $x, y \in X$ such that $x \nleq y$, there is a clopen upset U such that $x \in U$ and $y \notin U$;
(ii) If U is clopen, then also $U^{\downarrow}$ is clopen.

Given Esakia spaces X and Y, an *Esakia morphism* $p\colon X \to Y$ is a continuous map satisfying the two following conditions:

(i) For all $x, y \in X$ if $x \leq y$, then $p(x) \leq p(y)$;
(ii) For all $x \in X$ and $y \in Y$ such that $p(x) \leq y$, there exists $z \in X$ such that $x \leq z$ and $p(z) = y$.

Esakia duality is a dual categorical equivalence between the category of Heyting algebras with homomorphisms and the category of Esakia spaces with Esakia morphisms, which generalizes Stone duality. We shall review the definition of the two contravariant functors $(-)^*$ and $(-)_*$ witnessing Esakia duality. On the one hand, with every Heyting algebra H we associate an Esakia space H_* as follows. A *prime filter* F of H is a proper filter for which $x \vee y \in F$ entails $x \in F$ or $y \in F$. Then the Esakia space H_* is obtained by endowing the poset of prime filters of H ordered under the inclusion relation with the topology generated by the subbasis

$$\{\phi(a) \mid a \in H\} \cup \{\phi(a)^c \mid a \in H\},$$

where $\phi(a)$ is the set of prime filters of H containing a. Furthermore, every homomorphism $h : H \to H'$ between Heyting algebras is associated with the Esakia morphism $h_* : H'_* \to H_*$ defined as $h_*(F) = h^{-1}[F]$. On the other hand, with every Esakia space X we associate a Heyting algebra X^* as follows. Let $\mathsf{ClUp}(X)$ be the set of clopen upsets of X. Then

$$X^* = (\mathsf{ClUp}(X), \cap, \cup, \to, \emptyset, X),$$

where $\to$ is defined by letting $U \to V = ((U \smallsetminus V)^{\downarrow})^c$. Furthermore, every Esakia morphism $p : X \to Y$ is associated with the homomorphism $p^* : Y^* \to X^*$ defined as $p^*(U) = p^{-1}[U]$.

3. Poset Colourability

We will rely on the following concepts. The first item in the next definition exemplifies the idea of a *back and forth system*, and can also be seen as a version of the notion of layered bisimulation from [**visser1996uniform**]. The connection between back and forth systems and types goes back to Fraïssé [**fraisse1954quelques**].

Definition 1. Let X be a poset and $G \subseteq \mathsf{Up}(X)$.

(i) For every $n < \omega$ we define recursively an equivalence relation $\sim_n^G$ on X as follows: for every $x, y \in X$,

$$x \sim_0^G y \iff \forall g \in G\ (x \in g \iff y \in g);$$
$$x \sim_{n+1}^G y \iff \forall z \geq x\ \exists v \geq y\ (z \sim_n^G v)\ \wedge\ \forall v \geq y\ \exists z \geq x\ (z \sim_n^G v).$$

Moreover, we consider the following equivalence relation on X:

$$\sim_\omega^G = \bigcap_{n \in \omega} \sim_n^G.$$

(ii) The *n-type* and the *ω-type* over G of an element $x \in X$ are, respectively, the sets

$$\{y \in X \mid x \sim_n^G y\} \text{ and } \{y \in X \mid x \sim_\omega^G y\}.$$

(iii) We say an element $x \in X$ is *G-isolated* if $x \sim_\omega^G y$ entails $x = y$.

(iv) We say that X is *G-coloured* (or *coloured by* G) if every element of X is G-isolated.

Lemma 2. *Let X be a poset. For every $G \subseteq \mathsf{Up}(X)$ and $n < \omega$, the equivalence relation $\sim_{n+1}^G$ refines $\sim_n^G$.*

Proof. We proceed by induction. For $n = 0$ we assume $x \nsim_0^G y$. Without loss of generality there is some $g \in G$ such that $x \in g$ and $y \notin g$. Since g is an upset, it follows that $z \in g$ for every $z \geq x$ but $y \notin g$, showing $x \nsim_1^G y$.

For $n = m + 1$ assume $x \nsim_n^G y$. Without loss of generality there is some $z \geq x$ such that for all $v \geq y$ it holds $z \nsim_{n-1}^G v$. Hence, by the induction hypothesis, we obtain $z \nsim_n^G v$ and therefore $x \nsim_{n+1}^G y$. □

By a *term* we understand a first-order term in the language of Heyting algebras.

Definition 3. The *implication rank* $\mathsf{rank}(\phi)$ of a term ϕ is defined as follows:

(i) If ϕ is a constant or a variable, then $\mathsf{rank}(\phi) = 0$;
(ii) $\mathsf{rank}(\psi \wedge \chi) = \max\{\mathsf{rank}(\psi), \mathsf{rank}(\chi)\}$;
(iii) $\mathsf{rank}(\psi \vee \chi) = \max\{\mathsf{rank}(\psi), \mathsf{rank}(\chi)\}$;
(iv) $\mathsf{rank}(\psi \to \chi) = \max\{\mathsf{rank}(\psi), \mathsf{rank}(\chi)\} + 1$.

Let H be a Heyting algebra. Given a subset $G \subseteq H$, we denote by $\langle G \rangle$ the subalgebra of H generated by G. We recall that the universe of $\langle G \rangle$ is

$$\{\phi^H(\vec{g}) \mid \vec{g} \in G \text{ and } \phi \text{ is a term}\},$$

where $\phi^H(\vec{g})$ is the interpretation of ϕ in H under the assignment $\vec{g}$.

Definition 4. Let X be an Esakia space and $G = \{g_i \mid i < k\} \subseteq X^*$. The *implication rank* $\mathsf{rank}(U)$ of an element $U \in \langle G \rangle$ is

$$\min\{\mathsf{rank}(\phi) \mid \phi \text{ is a term such that } U = \phi^{X^*}(g_0, \ldots, g_{k-1})\}.$$

In addition, with every U as above we associate a term $\phi_U(x_0, \ldots, x_{k-1})$ such that $U = \phi_U^{X^*}(g_0, \ldots, g_{k-1})$ and $\mathsf{rank}(U) = \mathsf{rank}(\phi)$.

The following lemma and the subsequent Colouring Theorem generalize [**grilletti2023esakia**] and are essentially a reformulation of [**bezhanishvili2006lattices**]. The relation between the implication rank of a term and the existence of a back-and-forth system of corresponding length was established in [**visser1996uniform**].

Lemma 5. *Let X be an Esakia space and $x, y \in X$. The following condition holds for every finite $G \subseteq X^*$:*

$$x \sim_n^G y \iff \forall U \in \langle G \rangle \text{ with } \mathsf{rank}(U) \leq n : (x \in U \iff y \in U).$$

Proof. We fix an enumeration $G = \{g_i \mid i < k\}$ and let $\vec{g} = (g_0, \dots, g_{n-1})$. Moreover, given a term $\phi(x_0, \dots, x_{n-1})$, we will write $\phi(\vec{g})$ as a shorthand for $\phi^{X^*}(\vec{g})$. Both implications in the statement will be proven by induction on n.

($\Rightarrow$) For the case where $n = 0$, suppose that $x \sim_0^G y$ and consider $U \in \langle G \rangle$ such that $\mathsf{rank}(U) = 0$. Then we may assume that ϕ_U is a meet of joins of constants and variables. If $\phi_U = 0$ or $\phi_U = 1$, then $U = \emptyset$ or $U = X$ and the claim follows immediately. On the other hand, if $\phi_U = g_i$ for some $i < k$, we have that $x \in g_i$ if and only if $y \in g_i$ by the definition of $\sim_0^G$. As ϕ_U is a meet of joins of constants and variables, this implies that the claim holds.

Then we consider the case where $n = m+1$. Suppose that $x \sim_{m+1}^G y$ and consider $U \in \langle G \rangle$ such that $\mathsf{rank}(U) \leq m+1$. First suppose that $\mathsf{rank}(U) \leq m$. By Lemma 2 we have $x \sim_m^G y$ and, therefore, the claim holds by the induction hypothesis. Then we consider the case where $\mathsf{rank}(U) = m+1$. We may assume that ϕ_U is a conjunction of disjunctions of terms of the form $\alpha \to \beta$ of implication rank $\leq m+1$ and with variables among $x_0, \dots, x_{n-1}$.

For each of these implications $\alpha \to \beta$, let

$$U_{\alpha\beta} \coloneqq \alpha(\vec{g}) \to \beta(\vec{g}) = ((\alpha(\vec{g}) \smallsetminus \beta(\vec{g}))^{\downarrow})^c.$$

We will show that $x \in U_{\alpha\beta}$ if and only if $y \in U_{\alpha\beta}$. By symmetry, it suffices to prove the implication from right to left. Accordingly, suppose $x \notin ((\alpha(\vec{g}) \smallsetminus \beta(\vec{g}))^{\downarrow})^c$. Then there is some $z \geq x$ such that $z \in \alpha(\vec{g}) \smallsetminus \beta(\vec{g})$ and, since $x \sim_{m+1}^G y$, there is some $v \geq y$ such that $z \sim_m^G v$. As $\alpha(\vec{g}), \beta(\vec{g}) \in \langle G \rangle$ and $\mathsf{rank}(\alpha(\vec{g})), \mathsf{rank}(\alpha(\vec{g})) \leq m$, we can apply the induction hypothesis obtaining $v \in \alpha(\vec{g}) \smallsetminus \beta(\vec{g})$. Hence, we conclude that $y \notin ((\alpha(\vec{g}) \smallsetminus \beta(\vec{g}))^{\downarrow})^c = U_{\alpha\beta}$.

Since U is a meet of joins of sets of the form $U_{\alpha\beta}$, the claim follows from the fact no $U_{\alpha\beta}$ separates x and y.

($\Leftarrow$) For the case where $n = 0$, suppose that $x \nsim_0^G y$. Without loss of generality, we may assume that there is some $g_i \in G$ such that $x \in g_i$ and $y \notin g_i$. Since $\mathsf{rank}(g_i) = 0$, the claim follows immediately.

For the case where $n = m+1$, suppose that $x \nsim_{m+1}^G y$. We may assume that there is $z \geq x$ such that for all $v \geq y$ we have $z \nsim_m^G v$. By the induction hypothesis, for every $v \geq y$, there is either $\psi_v(\vec{g}) \in \langle G \rangle$ such that $z \in \psi_v(\vec{g})$ and $v \notin \psi_v(\vec{g})$, or $\chi_v \in \langle G \rangle$ such that $z \notin \chi_v(\vec{g})$ and $v \in \chi_v(\vec{g})$, with $\mathsf{rank}(\psi_v), \mathsf{rank}(\chi_v) \leq m$. We let

$$I_0 \coloneqq \{v \in y^{\uparrow} \mid z \in \psi_v(\vec{g}) \text{ and } v \notin \psi_v(\vec{g})\};$$
$$I_1 \coloneqq \{v \in y^{\uparrow} \mid z \notin \chi_v(\vec{g}) \text{ and } v \in \chi_v(\vec{g})\}.$$

By construction we have $y^{\uparrow} = I_0 \cup I_1$. Then we define

$$Z \coloneqq \bigcap_{v \in I_0} \psi_v(\vec{g}) \to \bigcup_{v \in I_1} \chi_v(\vec{g}) = ((\bigcap_{v \in I_0} \psi_v(\vec{g}) \smallsetminus \bigcup_{v \in I_1} \chi_v(\vec{g}))^{\downarrow})^c.$$

Notice that by the previous direction the number of terms of rank $\leq m$ is finite, whence the intersections and unions above are finitary and thus Z is a well-defined element of $\langle G \rangle$. Furthermore, $\mathsf{rank}(Z) \leq m+1$ because each ψ_v and χ_v has implication rank $\leq m$. Therefore, to conclude the proof, it suffices to show that $x \notin Z$ and $y \in Z$.

Since for every $v \in I_0$ we have $z \in \psi_v(\vec{g})$ and for every $v \in I_1$ we have $z \notin \chi_v(\vec{g})$, it follows that $z \in \bigcap_{v \in I_0} \psi_v(\vec{g}) \smallsetminus \bigcup_{v \in I_1} \chi_v(\vec{g})$. As $x \leq z$, we obtain

$$x \notin ((\bigcap_{v \in I_0} \psi_v(\vec{g}) \smallsetminus \bigcup_{v \in I_1} \chi_v(\vec{g}))^{\downarrow})^c = Z.$$

To prove that $y \in Z$, suppose the contrary. Then there is some $w \geq y$ such that $w \in \bigcap_{v \in I_0} \psi_v(\vec{g}) \smallsetminus \bigcup_{v \in I_1} \chi_v(\vec{g})$. As $w \geq y$ and $y^{\uparrow} = I_0 \cup I_1$, either $w \in I_0$ or $w \in I_1$.

If $w \in I_0$, then $w \notin \psi_w(\vec{g})$. While if $w \in I_1$, then $w \in \chi_w(\vec{g})$. In both cases, we obtain $w \notin \bigcap_{v \in I_0} \psi_v(\vec{g}) \smallsetminus \bigcup_{v \in I_1} \chi_v(\vec{g})$, a contradiction. □

Let X be an Esakia space and $G \subseteq X^*$. In view of Esakia duality, the subalgebra $\langle G \rangle$ of X^* is proper if and only if the relation

$$R = \{\langle x, y \rangle \in X \times X \mid x \in U \text{ iff } y \in U, \text{ for every } U \in \langle G \rangle\}$$

differs from the identity relation on X (see, e.g., [**bezhanishvili2006lattices**]). As a consequence, we deduce:

Lemma 6. *Let X be an Esakia space and $G \subseteq X^*$. Then $X^* = \langle G \rangle$ if and only if for every $x, y \in X$,*

$$\{U \in \langle G \rangle \mid x \in U\} = \{U \in \langle G \rangle \mid y \in U\} \text{ implies } x = y.$$

In view of the next result, the concept of subalgebra generation can be studied through that of colouring.

Colouring Theorem 7. *Let X be an Esakia space and $G \subseteq X^*$ finite. Then $X^* = \langle G \rangle$ if and only if X is G-coloured.*

Proof. ($\Rightarrow$) To prove that every element of X is G-isolated, it suffices to show that for every pair of distinct $x, y \in X$ we have $x \nsim^G_\omega y$. Accordingly, consider two distinct $x, y \in X$. By symmetry we may assume that $x \nleq y$. The Priestley separation axiom implies that there is $U \in X^*$ such that $x \in U$ and $y \notin U$. From the assumption that $X^* = \langle G \rangle$ it follows that $U \in \langle G \rangle$. By Lemma 5 we obtain that $x \nsim^G_n y$ for $n = \mathsf{rank}(U)$. Therefore, the definition of $\sim^G_\omega$ guarantees that $x \nsim^G_\omega y$.

($\Leftarrow$) By Lemma 6 it suffices to prove that if $x, y \in X$ are such that $\{U \in \langle G \rangle \mid x \in U\} = \{U \in \langle G \rangle \mid y \in U\}$, then $x = y$. Together with Lemma 5, the assumption that $\{U \in \langle G \rangle \mid x \in U\} = \{U \in \langle G \rangle \mid y \in U\}$ implies $x \sim^G_\omega y$. Since X is G-coloured, we conclude that $x = y$. □

4. The counterexamples

Our aim is to construct for each $n < \omega$ an infinite $(n+1)$-generated Heyting algebra whose n-generated subalgebras are of size $\leq m_n$ for some $m_n < \omega$. We will do this by exhibiting their dual Esakia spaces X_n.

Definition 8. For every $n < \omega$, let $\mathbb{X}_n = (X_n, \tau, \leq)$ be the ordered topological space where

$$X_n = \{x^l_i \mid l \leq 2^n \text{ and } i < \omega\} \cup \{x_\infty\},$$
$$\tau = \{U \in \mathcal{P}(X_n) \mid \text{if } x_\infty \in U, \text{ then } U \text{ is cofinite}\},$$

and $\leq$ is the unique partial order with minimum x_∞ such that for every $x^l_i, x^{l'}_{i'} \in X_n$,

$$x^l_i \leq x^{l'}_{i'} \iff \text{either } i \geq i' + 2 \text{ or } (i = i' + 1 \text{ and } l' \neq l + 1).$$

Lastly, for each $i < \omega$ we let $L^i_n = \{x^l_i \mid l \leq 2^n\}$ and we refer to this as the *i-th level/layer* of X_n.

Notice that X_0 is the dual of the Rieger-Nishimura lattice, i.e., the one-generated free Heyting algebra [**Ni60**, **Ri49**]. On the other hand, X_2 is depicted in Figure 1 as an exemplification.

Lemma 9. *For every $n < \omega$, X_n is an Esakia space.*

Proof. First, X_n is compact because it is the Alexandroff extension of the countable discrete space $X_n \smallsetminus \{x_\infty\}$. To prove that the Priestley separation axiom holds, consider $x, y \in X_n$ such that $x \nleq y$. Then x differs from the minimum x_∞. Consequently, $x^\uparrow$ is finite and omits x_∞. It follows that $x^\uparrow$ is a clopen upset which,

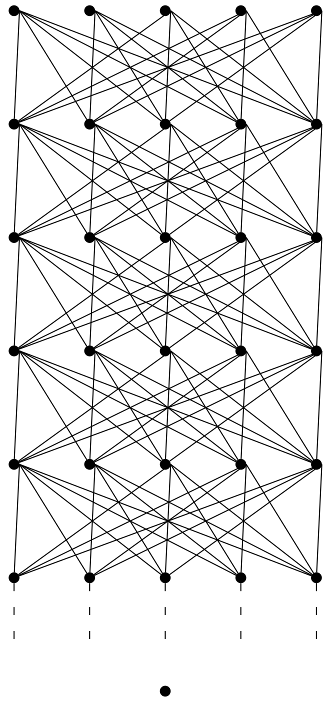

FIGURE 1. The Esakia space X_2

obviously, omits y. It only remains to prove that the downset of a nonempty clopen set U is clopen. Since U is open and nonempty, the definition of the topology guarantees that U contains an element of the form x_i^l. Therefore, $U^{\downarrow}$ contains $X_n \smallsetminus (L_n^0 \cup \cdots \cup L_n^{i+1})$. It follows that $U^{\downarrow}$ is cofinite and contains x_∞, whence it is clopen. □

Henceforth, we will identify natural numbers with finite ordinals, i.e., we identify each natural number $n < \omega$ with the set $\{m \in \omega \mid m < n\}$.

Definition 10. Let X be an Esakia space.

(i) A *colouring* of X is a function $c\colon n \to X^*$ where $n < \omega$ and X is coloured by $c[n]$;
(ii) X is said to be *n-colourable* if there is a colouring $c : n \to X^*$.

The following is an immediate consequence of the Colouring Theorem 7:

Proposition 11. *An Esakia space X is n-colourable if and only if X^* is n-generated.*

Consequently, in order to prove that X_n^* is $(n+1)$-generated, it suffices to show that X_n is $(n+1)$-colourable.

Proposition 12. *The Esakia space X_n is $(n+1)$-colourable.*

Proof. Since $n < n+1$ there is an injection $e : 2^n + 1 \to \wp(n+1)$. Then let $c : n+1 \to X_n^*$ be the map defined by letting

$$c(k) = \{x_0^l \in X_n \mid k \in e(l)\}.$$

By the definition of X_n we have

$$X_n = \{x_\infty\} \cup \bigcup_{i<\omega} L_n^i.$$

Therefore, to prove that X_n is $(n+1)$-colourable, it suffices to show that for every $i \in \omega$ the points in L_n^i are all $c[n+1]$-isolated. We proceed by induction on i, noting that, by the definition of c and of $\sim_0^{c[n+1]}$, it is clear that every point in L_n^0 is $c[n+1]$-isolated. Now, let $i > 0$ and assume that, for all $j < i$, every point in L_n^j is $c[n+1]$-isolated.

Let us first show that $x_i^l \not\sim_\omega^{c[n+1]} x_i^{l'}$, for every $l \neq l' \leq 2^n$. By the construction of X_n we can suppose, without loss of generality, that there exists $z \in L_n^{i-1}$ lying above x_i^l but not above $x_i^{l'}$. As z is $c[n+1]$-isolated by our induction hypothesis, it follows that for all $v \geq x_i^{l'}$, there exists m_v satisfying $z \not\sim_{m_v}^{c[n+1]} v$. Take $m \coloneqq \max\{m_v \in \omega \colon v \geq x_i^{l'}\}$, which exists, as $(x_i^{l'})^{\uparrow}$ is finite. By Lemma 2 we have that $z \not\sim_m^{c[n+1]} v$ for every $v \geq x_i^{l'}$. It is now clear that $x_i^l \not\sim_{m+1}^{c[n+1]} x_i^{l'}$, thus $x_i^l \not\sim_\omega^{c[n+1]} x_i^{l'}$.

Next we show that, given $l \leq 2^n$, then for every $k > i$ and $y \in L_n^k$, we have $x_i^l \not\sim_\omega^{c[n+1]} y$. By the construction of X_n we know that $x_i^{l'} > y$, for some $l \neq l' \leq 2^n$. It follows from our induction hypothesis and from what we just proved above that for every $z \geq x_i^l$, there exists $m_z \in \omega$ such that $z \not\sim_{m_z}^{c[n+1]} x_i^{l'}$. As $(x_i^l)^{\uparrow}$ is finite, taking $m' \coloneqq \max\{m_z \in \omega \colon z \geq x_i^l\}$ and applying Lemma 2 yields $z \not\sim_{m'}^{c[n+1]} x_i^{l'}$ for every $z \geq x_i^l$. Since $x_i^{l'} \geq y$, this implies $x_i^l \not\sim_{m'+1}^{c[n+1]} y$, hence also $x_i^l \not\sim_\omega^{c[n+1]} y$.

Again using our induction hypothesis, we can now conclude that every point in L_n^i is $c[n+1]$-isolated. □

Corollary 13. *The Heyting algebra X_n^* is infinite and $(n+1)$-generated.*

Proof. Since X_n is infinite, the Heyting algebra X_n^* is also infinite. Furthermore, it is $(n+1)$-generated by Propositions 11 and 12. □

Therefore, it only remains to prove that the n-generated subalgebras of X_n^* are of size $\leq m_n$ for some $m_n < \omega$. The proof of this fact will be based on the next three technical lemmas.

Lemma 14. *Let $c : m \to X_n^*$ be a function and $i, k < \omega$ such that the following conditions hold:*

(i) $|L_n^i / {\sim_\omega^{c[m]}}| \leq k \leq 2^n$*;*

(ii) For all $x, y \in L_n^{i+1}$ we have $x \sim_0^{c[m]} y$.

Then $|L_n^{i+1} / {\sim_\omega^{c[m]}}| \leq k \leq 2^n$.

Proof. By condition (i) we can enumerate $L_n^i / {\sim_\omega^{c[m]}}$ as $\{A_j \mid j < k\}$. We say that an element $x \in L_n^{i+1}$ sees some A_j when $x \in A_j^{\downarrow}$.

Claim 15. *If two elements of L_n^{i+1} see the same A_j's, then they have the same ω-type.*

Proof of the Claim. Consider $x, y \in L_n^{i+1}$ and suppose that they see the same A_j's. We need to show that $x \sim_p^{c[m]} y$ for every $p < \omega$. The proof proceeds by induction on p. The case where $p = 0$ holds by condition (ii). For the case where $p = q + 1$, the induction hypothesis guarantees that $x \sim_q^{c[m]} y$. Then consider some $z \geq x$. We need to find some $v \geq y$ such that $z \sim_q^{c[m]} v$. If $z = x$, then we are done taking $v \coloneqq y$. Then we consider the case where $x < z$. If $z \in (L_n^{i-1})^{\uparrow}$, the definition of X_n and the assumption that $x, y \in L_n^{i+1}$ guarantee that $y \leq z$, in which case we take $v \coloneqq z$. It only remains to consider the case where $z \in L_n^i$. Clearly, there exists $j < k$ such that $z \in A_j$. Therefore, x sees A_j and so does y by assumption. Let $v \in A_j$ be such that $y \leq v$. As $z, v \in A_j$, the elements z and v have the same ω-type, whence $z \sim_q^{c[m]} v$. □

Now, observe that if A_j contains at least 2 elements of L_n^i, then every element of L_n^{i+1} sees A_j because of the structure of X_n. Furthermore, as L_n^i has 2^n+1 elements and the A_j's are exactly $k \leq 2^n$, we may assume without loss of generality that A_{k-1} contains at least 2 elements of L_n^i. Together with the Claim, this implies that if two elements of L_n^{i+1} see the same elements of $\{A_j \mid j < k-1\}$, then they have the same ω-type. As the structure of X_n guarantees that every element of L_n^{i+1} sees every A_j except possibly one, we conclude that $L_n^{i+1}/{\sim_\omega^{c[m]}}$ has $\leq k$ elements. $\square$

Lemma 16. *For every function $c \colon n \to X_n^*$, there is $j < \omega$ such that at least two elements in L_n^j have the same ω-type and every element in $(L_n^{j+1})^{\downarrow}$ has the same 0-type.*

Proof. For each $l < n$, let $U_l := c(l)$. We may assume without loss of generality that every U_l is finite and nonempty, for otherwise the topology of X_n would yield $U_l = \emptyset$ or $U_l = X_n$, in which case U_l does not contribute to distinguish between the ω-type or 0-type of the elements of X_n. Furthermore, for each $l < n$ we denote by i_l the least i such that $U_l \cap L_n^{i+1} = \emptyset$, which exists because U_l is finite. Lastly, we may assume without loss of generality that $i_l \leq i_{l'}$ for each $l < l'$.

Claim 17. *For each $l < n$, the number of distinct ω-types over $c[l+1]$ of the elements of $L_n^{i_l}$ is bounded above by 2^{l+1}. Furthermore, every member of $(L_n^{i_l+1})^{\downarrow}$ has the same 0-type over $c[l+1]$.*

Proof of the Claim. From the definition of $i_0, \dots, i_l$ and the assumption that $i_0 \leq \dots \leq i_l$ it follows that $(L_n^{i_l+1})^{\downarrow} \cap (U_0 \cup \dots \cup U_l) = \emptyset$. Therefore, the last part of the claim holds. We prove the first part of the claim by induction on l.

Induction Base. We need to prove that the number of distinct ω-types over $c[1] = \{U_0\}$ of the elements of $L_n^{i_0}$ is ≤ 2. It suffices to show that

$$(x \in U_0 \iff y \in U_0) \text{ implies } x \sim_\omega^{c[1]} y,$$

for every pair of distinct $x, y \in L_n^{i_0}$. If $i_0 = 0$, this is clear, as the sole possible ω-types over $\{U_0\}$ of the elements of L_n^0 are U_0 and $\{x \in X_n \mid x^{\uparrow} \cap U_0 = \emptyset\}$. Then we consider the case where $i_0 > 0$. Clearly, if $x, y \in U_0$, the ω-type over $\{U_0\}$ of x and y is U_0. Then we consider the case where $x, y \notin U_0$. If $L_n^{i_0-1} \subseteq U_0$, then $x, y \notin U_0$, but $x^{\uparrow} \smallsetminus \{x\}, y^{\uparrow} \smallsetminus \{y\} \subseteq U_0$, so that $x \sim_\omega^{c[1]} y$. Then we consider the case where $L_n^{i_0-1} \nsubseteq U_0$. By assumption there is an element $z \in U_0 \cap L_n^{i_0}$. The definition of X_n guarantees that the set $z^{\uparrow} \cap L_n^{i_0-1}$ is either $L_n^{i_0-1}$ or $L_n^{i_0-1} \smallsetminus \{v\}$ for some $v \in X_n$. Since $z^{\uparrow} \subseteq U_0$ and $L_n^{i_0-1} \nsubseteq U_0$, we obtain $U_0 \cap L_n^{i_0-1} = L_n^{i_0-1} \smallsetminus \{v\}$ for some $v \in X_n$. As x and y are distinct from z (because $x, y \notin U_0$ and $z \in U_0$) and $z \nleq v \in L_n^{i_0-1}$, the definition of X_n guarantees that $x, y < v$. As $x, y, v \notin U_0$ and $x^{\uparrow} \smallsetminus \{x, v\}, y^{\uparrow} \smallsetminus \{y, v\} \subseteq z^{\uparrow} \subseteq U_0$, we conclude that $x \sim_\omega^{c[1]} y$ as desired.

Induction Step. Suppose that the statement holds for l, i.e., that the number of distinct ω-types over $c[l+1]$ of the elements of $L_n^{i_l}$ is bounded above by 2^{l+1}. We will prove that also holds when l is replaced by $l+1$. If $i_{l+1} = 0$, this is clear, as the sole possible ω-types over $\{U_0, \dots, U_{l+1}\}$ of the elements of $L_n^{i_{l+1}} = L_n^0$ are the sets of the form $\bigcap_{j \in J} U_j \cap \bigcap_{j \notin J} U_j^c$ for some $J \subseteq l+1$. Thus we may assume that $i_{l+1} > 0$.

We will prove that the number of distinct ω-types over $c[l+1]$ of the elements of $L_n^{i_{l+1}-1}$ is at most 2^{l+1}, that is,

$$|L_n^{i_{l+1}-1}/{\sim_\omega^{c[l+1]}}| \leq 2^{l+1}. \tag{1}$$

We have two cases: either $i_l < i_{l+1}$ or $i_l = i_{l+1}$. Suppose first that $i_l < i_{l+1}$. If $i_l = i_{l+1} - 1$ we are done by the inductive assumption. Then we may assume that $i_l < i_{l+1} - 1$. Recall that $i_0 \leq \cdots \leq i_l < i_l + 1 \leq i_{l+1}$. Moreover, from the definition of $i_0, \ldots, i_l$ it follows

$$(U_0 \cup \cdots \cup U_l) \cap (L_n^{i_l+1})^{\downarrow} = \emptyset.$$

Therefore, for each $x, y \in (L_n^{i_l+1})^{\downarrow}$ it holds $x \sim_0^{c[l+1]} y$. Consequently, the result follows from the induction hypothesis and $i_{l+1} - i_l$ applications of Lemma 14. It only remains to consider the case where $i_l = i_{l+1}$.

Let m be the greatest integer such that $i_m < i_l = i_{l+1}$, if it exists, or -1 otherwise. If $m \neq -1$, then $m < l$ and by the induction hypothesis, the number of distinct ω-types over $c[m+1]$ of the elements of $L_n^{i_m}$ is at most 2^{m+1}. Since

$$(U_0 \cup \cdots \cup U_{i_m}) \cap (L_n^{i_m+1})^{\downarrow} = \emptyset,$$

by the definition of the i_t's and by the structure of X_n, it follows from $i_l - 1 - i_m$ applications of Lemma 14 that $|L_n^{i_l-1}/{\sim_\omega^{c[m+1]}}| \leq 2^{m+1}$.

Now, for an arbitrary m satisfying the above definition, notice that if $m < k < l + 1$, then $i_k = i_l = i_{l+1}$. Again by the definition of the i_t's, it follows that, for any such k, we have that $L_n^j \subseteq U_k = c(k)$, for every $j < i_l - 1$. As $m < m + 1 < l + 1$ because $m < l$, the ω-type of an element x of $L_n^{i_l-1}$ over $c[l+1]$ is totally determined by its ω-type over $c[m+1]$ (of which there are none if $m = -1$, and at most 2^{m+1} otherwise, by above) together with whether or not x belongs to U_k, for each $m < k < l + 1$. Thus, there are at most $2^{m+1} \cdot 2^{l-m} = 2^{l+1}$ possible ω-types over $c[l+1]$ that $x \in L_n^{i_l-1}$ can have, as desired. This concludes the proof of condition (1).

To conclude the proof of the claim, it is convenient to separate the following cases.

Case A. Suppose that $L_n^{i_{l+1}-1} \subseteq U_{l+1}$. This entails that the ω-types over $c[l+2]$ of the elements in $L_n^{i_{l+1}-1}$ are the same as those over $c[l+1]$. Hence, the elements from $L_n^{i_{l+1}}$ which see the same ω-types over $c[l+1]$ from $L_n^{i_{l+1}-1}$ also see the same ω-types over $c[l+2]$ from $L_n^{i_{l+1}-1}$. Consequently, the ω-types over $c[l+2]$ of the elements in $L_n^{i_{l+1}}$ are determined by their ω-types over $c[l+1]$ (of which there are at most 2^{l+1}, by induction hypothesis and by possibly repeatedly applying Lemma 14 if $i_l < i_{l+1}$) together with whether or not they belong to the set $U_{l+1} \cap L_n^{i_{l+1}}$, in the sense that for $x, y \in L_n^{i_{l+1}}$,

$$x \sim_\omega^{c[l+2]} y \iff x \sim_\omega^{c[l+1]} y \text{ and } x \sim_0^{\{U_{l+1}\}} y.$$

This gives us at most 2^{l+2} possible ω-types over $c[l+2]$ for elements of $L_n^{i_{l+1}}$.

Case B. Suppose that $L_n^{i_{l+1}-1} \nsubseteq U_{l+1}$. Recall that $U_{l+1} \cap L_n^{i_{l+1}} \neq \emptyset$ by the definition of i_{l+1}. Moreover, by the definition of X_n the upset generated by any pair of distinct elements of $L_n^{i_{l+1}}$ contains the whole $L_n^{i_{l+1}-1}$. Therefore, the assumption that $L_n^{i_{l+1}-1} \nsubseteq U_{l+1}$ allows us to assume, without loss of generality, that $L_n^{i_{l+1}} \cap U_{l+1} = \{x_{i_{l+1}}^0\}$ and $L_n^{i_{l+1}-1} \cap U_{l+1} = L_n^{i_{l+1}-1} \smallsetminus \{x_{i_{l+1}-1}^1\}$. By condition (1), the elements in $L_n^{i_{l+1}-1} \cap U_{l+1} = \{x_{i_{l+1}-1}^t \mid t \neq 1\}$ have at most 2^{l+1} different ω-types over $c[l+1]$ and, since they all belong to U_{l+1}, they have at most 2^{l+1} different ω-types also over $c[l+2]$.

Now, the definition of X_n guarantees that for every $m > 0$ we have that $x_{i_{l+1}}^m \leq x_{i_{l+1}-1}^1$. Furthermore, $x_{i_{l+1}}^m$ does not belong to U_{l+1} because $L_n^{i_{l+1}} \cap U_{l+1} = \{x_{i_{l+1}}^0\}$. Therefore, the ω-type over $c[l+2]$ of an element of the form $x_{i_{l+1}}^m$ with $m > 0$ is determined by the fact that $x_{i_{l+1}}^m$ does not belong to U_{l+1} and by the elements of $L_n^{i_{l+1}-1}$ with distinct ω-types over $c[l+2]$ it sees. Since $x_{i_{l+1}}^m$ sees all but possibly one of the element of $L_n^{i_{l+1}-1}$ and the elements of $L_n^{i_{l+1}-1}$ have at most 2^{l+1} different

ω-types, this implies that

$$|L_n^{i_{l+1}} \smallsetminus \{x^0_{i_{l+1}}\}/{\sim_\omega^{c[l+2]}}| \leq 2^{l+1} + 1.$$

Consequently,

$$|L_n^{i_{l+1}}/{\sim_\omega^{c[l+2]}}| \leq 2^{l+1} + 2 \leq 2^{l+2},$$

thus finishing the proof of the claim. □

From the Claim it follows that the number of ω-types of the elements in $L_n^{i_{n-1}}$ over $c[n]$ is bounded above by 2^n. Since $L_n^{i_{n-1}}$ has $2^n + 1$ elements, it follows that at least two elements of $L_n^{i_{n-1}}$ have the same ω-type. Moreover, the second part of the Claim guarantees that every element of $(L_n^{i_{n-1}+1})^{\downarrow}$ has the same 0-type over $c[n]$. Thus, the statement holds for $j \coloneqq i_{n-1}$. □

Lemma 18. *Let $c\colon m \to X_n^*$ be a function and $i < \omega$. Suppose two distinct elements of L_n^i have the same ω-type over $c[m]$ and that for every $q > i$ the elements of L_n^q have the same 0-type over $c[m]$. Then every element of $(L_n^{i+2^n+1})^{\downarrow}$ has the same ω-type over $c[m]$.*

Proof. Since two distinct elements of L_n^i have the same ω-type over $c[m]$ and all the elements of L_n^{i+1} have the same 0-type over $c[m]$, then it follows from the construction of X_n that $x^{2^n}_{i+1} \sim_\omega^{c[m]} x^j_{i+1}$ for some $j < 2^n$. Therefore, the construction of X_n guarantees that the elements $x^{2^n-1}_{i+2}$ and $x^{2^n}_{i+2}$ see the same equivalence classes of $L_n^{i+1}/{\sim_\omega^{c[m]}}$. Since by assumption $x^{2^n-1}_{i+2}$ and $x^{2^n}_{i+2}$ have the same 0-type over $c[m]$, this implies that $x^{2^n-1}_{i+2} \sim_\omega^{c[m]} x^{2^n}_{i+2}$. For the same reason, we have that $x^{2^n-2}_{i+3} \sim_\omega^{c[m]} x^{2^n-1}_{i+3} \sim_\omega^{c[m]} x^{2^n}_{i+3}$. By proceeding in this way, we obtain that every element of $L_n^{i+2^n+1}$ has the same ω-type over $c[m]$. Since for $t \geq i + 2^n + 1$ every element of L_n^t has the same 0-type over $c[m]$, this is enough to conclude that every element of $(L_n^{i+2^n+1})^{\downarrow}$ has the same ω-type over $c[m]$. □

We are now ready to prove that the Esakia space X_n satisfies the desired property:

Proposition 19. *There exists $m_n < \omega$ such that the n-generated subalgebras of X_n^* are of size $\leq m_n$.*

Proof. We begin by the following observation.

Claim 20. *There exists $k < \omega$ such that for every function $c\colon n \to X_n^*$ the number of ω-types over $c[n]$ of elements of X_n is $\leq k$.*

Proof of the Claim. Recall the definition of the integers $i_0, \dots, i_{n-1}$ associated with c in the proof of Lemma 16. In view of Claim 17 and the fact that each $L_n^{i_l}$ has $2^n + 1$ elements, we obtain that for each i_l at least two elements of $L_n^{i_l}$ have the same ω-type over $c[l+1]$ and every point in $(L_n^{i_l+1})^{\downarrow}$ has the same 0-type over $c[l+1]$. Hence, it follows from Lemma 18 that all the elements of $(L_n^{i_l+2^n+1})^{\downarrow}$ have the same ω-type over $c[l+1]$. Furthermore, from the construction of X_n and the definition of i_l it follows that the upset $(L_n^{i_l-2})^{\uparrow}$ (which is the emptyset if $i_l \leq 1$) is contained in $U_l \coloneqq c(l)$. Let us partition X_n as the union

$$(L_n^{i_l-2})^{\uparrow} \cup L_n^{i_l-1} \cup L_n^{i_l} \cup \dots \cup L_n^{i_l+2^n} \cup (L_n^{i_l+2^n+1})^{\downarrow},$$

and note that the above discussion entails that the effect of the clopen U_l in the determination of the ω-type over $c[n]$ of a point x is trivial if $x \in (L_n^{i_l-2})^{\uparrow} \cup (L_n^{i_l+2^n+1})^{\downarrow}$, and noticeable only if $x \in L_n^{i_l-1} \cup L_n^{i_l} \cup \dots \cup L_n^{i_l+2^n}$. Since each of these $(i_l + 2^n) - (i_l - 2) = 2^n + 2$ layers has $2^n + 1$ many elements, we conclude that the clopen U_l can only contribute to distinguish at most $(2^n+1)(2^n+2)+2$ ω-types over $c[n]$ in X_n.

Since the i_l in the above argument was arbitrary, it now follows that each clopen in $c[n] = \{U_0, \ldots, U_{n-1}\}$ can only contribute to distinguish at most $(2^n+1)(2^n+2)+2$ ω-types over $c[n]$ in X_n. Therefore, there are at most $k \coloneqq n[(2^n+1)(2^n+2)+2]$ distinct ω-types over $c[n]$ in X_n. As this bound is independent of the choice of the function c, we have found the desired uniform upper bound. □

In order to conclude the proof, it suffices to show that the n-generated subalgebras of X_n^* are of size $\leq 2^k$, for in this case the statement holds for $m_n \coloneqq 2^k$. Suppose, on the contrary, that there is an n-generated subalgebra of X_n^* containing distinct elements $U_0, \ldots, U_{2^k}$. Moreover, let $m < \omega$ be such that $\mathsf{rank}(U_i) \leq m$ for every $i \leq 2^k$. Since every family of 2^k+1 distinct subsets of a set Y separates at least $k+1$ elements of Y, there are distinct $x_0, \ldots, x_k \in X_n$ that are separated by $U_0, \ldots, U_{2^k}$. By Lemma 5 the elements $x_0, \ldots, x_k$ are unrelated by $\sim_m^{c[n]}$. By the definition of $\sim_\omega^{c[n]}$, this implies that $x_0, \ldots, x_k$ are also unrelated by $\sim_\omega^{c[n]}$. Therefore, there are $k+1$ distinct ω-types over $c[n]$ (that is, $x_0/{\sim_\omega^{c[n]}}, \ldots, x_k/{\sim_\omega^{c[n]}}$), but this contradicts Claim 20. □

5. The main result

Given a class K of similar algebras, we let

$$\mathbb{V}(\mathsf{K}) \coloneqq \text{the variety generated by } \mathsf{K};$$
$$\mathbb{S}(\mathsf{K}) \coloneqq \text{the class of subalgebras of the members of } \mathsf{K}.$$

The aim of this section is to establish the main result of the paper, namely:

Theorem 21. *For each $n < \omega$, the variety $\mathbb{V}(X_n^*)$ is strictly n-finite, i.e., it is n-finite but contains an infinite $(n+1)$-generated algebra.*

In particular, the variety $\mathbb{V}(X_n^*)$ is n-finite, but not locally finite. For the case where $n = 2$, this provides a negative answer to [**bezhanishvili2005locally**]. The proof of Theorem 21 relies on the next observation [**Be05k**].

Proposition 22. *Let K be a variety and Var a set of variables. Moreover, let $\{v_i \mid i \in I\}$ be a family of functions $v_i \colon Var \to A_i$ with $A_i \in \mathsf{K}$ such that for every pair of terms φ and ψ with variables in Var it holds that*

$$\mathsf{K} \nvDash \varphi \approx \psi \text{ implies that there exists } i \in I \text{ such that } \varphi^{A_i}(v_i(\vec{x})) \neq \psi^{A_i}(v_i(\vec{x})).$$

Then the n-generated free algebra of K embeds into the direct product $\prod_{i \in I} A_i$.

As a consequence, we deduce:

Corollary 23. *Let K be a class of similar algebras of finite type and $n < \omega$. If the cardinality of the n-generated members of $\mathbb{S}(\mathsf{K})$ is bounded above by some $m_n < \omega$, then $\mathbb{V}(\mathsf{K})$ is n-finite.*

Proof. Since the type of K is finite and the cardinality of the n-generated members of $\mathbb{S}(\mathsf{K})$ is bounded above by some $m_n < \omega$, up to isomorphism there are only finitely many n-generated algebras in $\mathbb{S}(\mathsf{K})$, all of whom are finite. We enumerate them as $H_0, \ldots, H_k$.

Now, consider the set of variables $Var \coloneqq \{x_0, \ldots, x_{n-1}\}$. Clearly, if two terms φ and ψ with variables in Var differ when interpreted in the variety $\mathbb{V}(\mathsf{K})$, then there exist some $i \leq k$ and a function $v \colon Var \to H_i$ such that $\varphi^{H_i}(v(\vec{x})) \neq \psi^{H_i}(v(\vec{x}))$. Therefore, we can apply Proposition 22 obtaining that the free n-generated algebra F_n of $\mathbb{V}(\mathsf{K})$ embeds into $H \coloneqq H_0^{Var} \times \cdots \times H_k^{Var}$. Since both Var and each H_i

are finite, so is H and, therefore, F_n. As every n-generated member of $\mathbb{V}(\mathsf{K})$ is a homomorphic image of F_n, we conclude that $\mathbb{V}(\mathsf{K})$ is n-finite. □

We are now ready to prove Theorem 21.

Proof. By Corollary 13 the Heyting algebra X_n^* is infinite and $(n+1)$-generated. As $X_n^* \in \mathbb{V}(X_n^*)$, it only remains to prove that the variety $\mathbb{V}(X_n^*)$ is n-finite. But this is an immediate consequence of Proposition 19 and Corollary 23. □

Remark 24. In view of Theorem 21, for each $n < \omega$ there is an n-finite variety of Heyting algebras that contains an infinite $(n+1)$-generated algebra. We will prove that such a variety can be chosen finitely axiomatisable.

First, recall from Theorem 21 that $\mathbb{V}(X_n^*)$ is n-finite. Therefore, there is some $m < \omega$ such that every n-generated member of $\mathbb{V}(X_n^*)$ has size $\leq m$ (for instance, m can be taken to be the size of the free n-generated algebra of $\mathbb{V}(X_n^*)$). This property can be expressed by a first-order sentence, namely,

$$\theta_n = \forall (x_i)_{i<n} \exists (y_j)_{j<m} \, ((\bigwedge_{i<n} x_i = y_i) \wedge (\bigvee_{j<m} y_j = 0) \wedge (\bigvee_{j<m} y_j = 1) \wedge \\ (\bigwedge_{\odot \in \{\wedge, \vee, \to\}} \bigwedge_{i,i'<m} \bigvee_{j<m} y_i \odot y_{i'} = y_j)\Big).$$

Then let Σ_n be the set of universally quantified equations valid in $\mathbb{V}(X_n^*)$. Since $\mathbb{V}(X_n^*)$ is a variety, it is axiomatized by Σ_n. Together with the fact that every n-generated member of $\mathbb{V}(X_n^*)$ has size $\leq m$, this implies $\Sigma_n \models \theta_n$ (where $\models$ stands for the consequence relation of first-order logic). By the compactness theorem there is a finite $T \subseteq \Sigma_n$ such that $T \models \theta_n$. Let $\mathcal{V}$ be the class of all the Heyting algebras satisfying T. Clearly, $\mathcal{V}$ is a variety of Heyting algebras. Furthermore, from $T \subseteq \Sigma_n$ it follows immediately that $X_n^* \in \mathcal{V}$. Therefore, $\mathcal{V}$ contains an infinite $(n+1)$-generated algebra, namely, X_n^* (Corollary 13). Lastly, $\mathcal{V}$ is n-finite because $T \models \theta_n$ and, therefore, every n-generated algebra in $\mathcal{V}$ is of size $\leq m$.

Remark 25. From a logical standpoint, the importance of Heyting algebras is that they algebraize the *intuitionistic propositional calculus* IPC in the sense of [**BP89**]. As a consequence, the axiomatic extensions of IPC (known as *superintuitionistic logics*, or si-logics for short) form a lattice that is dually isomorphic to that of varieties of Heyting algebras (see, e.g., [**ChZa97**]). Because of this, Remark 24 can be rephrased as follows: for every $n < \omega$ there is a finitely axiomatisable si-logic that has only finitely many formulas in variables $x_0, \dots, x_{n-1}$ up to logical equivalence, but that has infinitely many nonequivalent formulas in variables $x_0, \dots, x_n$.

T. Hyttinen: Department of Mathematics and Statistics, University of Helsinki, P.O. Box 68 (Pietari Kalmin katu 5), 00014 Helsinki, Finland
Email address: tapani.hyttinen@helsinki.fi

M. Martins: Departament de Filosofia, Facultat de Filosofia, Universitat de Barcelona (UB), Carrer Montalegre, 6, 08001 Barcelona, Spain
Email address: miguel.pedrosodelima@ub.edu

T. Moraschini: Departament de Filosofia, Facultat de Filosofia, Universitat de Barcelona (UB), Carrer Montalegre, 6, 08001 Barcelona, Spain
Email address: tommaso.moraschini@ub.edu

D. E. Quadrellaro: Department of Mathematics and Statistics, University of Helsinki, P.O. Box 68 (Pietari Kalmin katu 5), 00014 Helsinki, Finland
Email address: davide.quadrellaro@helsinki.fi